Abstract

In this thesis, we establish a data-based framework for verifying control-theoretic properties and synthesising state-feedback controllers for nonlinear systems. Since a precise model of the underlying dynamics is not required a priori, this framework circumvents a time-consuming modelling via first principles. More precisely, we introduce a novel data-based system representation by combining *polynomial approximation* techniques together with the *set-membership* methodology. Through *robust control* methods, bounds on the noise and the error using polynomial approximation are taken into account. Thereby, a data-driven system analysis and state-feedback design with rigorous guarantees are achieved. Moreover, the polynomial characterization of the proposed system representation facilitates solving these control-related problems by means of SDPs from SOS relaxation even though the unknown dynamics are nonlinear. The thesis is separated into two parts. First, we investigate the determination of various input-output properties for the special class of nonlinear systems with polynomial dynamics from recorded input-state trajectories corrupted by noise. This constitutes the basis for the development of the data-driven polynomial system representation for nonlinear systems in the second part.

Data-based system analysis of polynomial systems

Chapter 3 addresses the verification of dissipativity for unknown polynomial discrete-time systems from noisy input-state data. Moreover, the determination of IQCs including a simultaneous optimization of the linear filter is addressed. For this purpose, we construct the set of all polynomial systems consistent with the observed data. More specifically, distinct non-conservative and over-approximating formulations for this set of systems are provided to attain either tighter or computationally cheaper characterizations. Given the knowledge of upper bounds on the noise and the degree of the polynomials of the dynamics, these sets contain the true system. By robust control techniques and a SOS relaxation, we can directly verify whether all systems of the feasible system set, including the true system, satisfy the corresponding input-output property by means of solving an SDP. Thereby, we can ensure the satisfaction of the system property from noisy data even though a precise model of the system is not identified. This finding, among others, is supplemented by combining the purely data-driven approach with prior model knowledge and a theoretical examination of the asymptotic consistency for an infinite amount of data.

Data-driven control of nonlinear systems by polynomial approximation

Based on the previous chapter and polynomial approximation, Chapter 4 introduces a novel data-based system representation of nonlinear systems tailored for system analysis and state-feedback design by SDPs. To this end, well-investigated bounds on the approximation error by polynomial interpolation are combined with a data-based set membership for the unknown interpolation polynomial. In particular, we focus on TPs as well as HPs, which offer additional degrees of freedom to improve the polynomial approximation. To further refine the accuracy of the system representation, we propose to obtain multiple polynomial approximations from data, where each approximation only needs to represent the dynamics in a subset of the operation set. Since the obtained system representation is described by polynomial sectors, the subsequent determination of input-output properties can be executed by SDPs from SOS relaxation. In addition to Chapter 3, a state-feedback design with performance criteria and the case of Gaussian noise are studied. The findings are evaluated in numerical examples and in experiments on a two-tank system.

In conclusion, this thesis establishes a framework for the verification of control-theoretic properties and the synthesis of state-feedback controllers for polynomial and nonlinear dynamical systems from noisy input-state data. Despite unknown nonlinearities and noise-corrupted data, the proposed set-membership approach achieves rigorous guarantees and enables an effective solution of the control-related problems by means of SDPs.

Deutsche Kurzfassung

Diese Doktorarbeit befasst sich mit der Verifikation systemtheoretischer Eigenschaften und der Synthese von Zustandsrückführungen für nichtlineare dynamische Systeme auf Grundlage verrauschter Daten. Da kein präzises Modell der zugrundeliegenden Dynamik im Vorfeld benötigt wird, umgeht der präsentierte Ansatz eine zeitaufwendige Modellierung durch physikalische Grundgesetze. Dazu führen wir eine neuartige datenbasierte Systembeschreibung ein, indem Techniken der *polynomiellen Approximation* und aus der *set-membership* Literatur kombiniert werden. Durch die Einbeziehung des Messrauschens und des Fehlers der Polynomapproximation mittels *robuster Regelungstechniken* werden rigorose Garantien für die Verifikation von Dissipativitätseigenschaften und den Entwurf von Zustandsrückführungen sichergestellt. Aufgrund der polynomiellen Charakterisierung der datenbasierten Systembeschreibung führen diese regelungstechnischen Problemstellungen durch SOS Relaxationen zu SDPs, obwohl die unbekannte Dynamik nichtlinear ist. Die Doktorarbeit unterteilt sich in zwei Hauptteile. Zuerst werden Rückschlüsse auf diverse Eingangs-/Ausgangseigenschaften aus gemessenen Eingangs-/Zustandsdaten für den Spezialfall von nichtlinearen Systemen mit polynomieller Dynamik untersucht. Dieses Ergebnis bildet die Basis für die datenbasierte polynomielle Systembeschreibung für nichtlineare Systeme im zweiten Teil.

Datengetriebene Systemanalyse polynomieller Systeme

Kapitel 3 adressiert den Nachweis von Dissipativität und IQCs, mit einer gleichzeitigen Optimierung des linearen Filters, für unbekannte polynomielle zeitdiskrete Systeme aus verrauschten Eingangs-/Zustandsdaten. Dafür konstruieren wir zuerst die Menge aller polynomieller Systeme, die die beobachteten Daten erklären können. Um entweder eine präzise oder rechentechnisch günstige Charakterisierung dieser Menge zu erhalten, präsentieren wir verschiedene nicht-konservative und überapproximierende Formulierungen. Falls obere Schranken an das Rauschen und den Grad der Dynamikpolynome bekannt ist, enthalten diese Systemmengen die tatsächliche Dynamik. Mittels robuster Regelungstechniken und SOS Relaxierung kann durch Lösen von SDPs direkt überprüft werden, ob alle enthaltenen Systeme, einschließlich dem wahren System, die Eingangs-/Ausgangseigenschaft erfüllen. Dadurch können wir aus verrauschten Daten sicherstellen, dass die Systemeigenschaft erfüllt ist, obwohl kein genaues Systemmodell identifiziert wurde. Dieser datengetriebene Ansatz wird durch zusätzliches Modellwissen und eine theoretische Untersuchung der asymptotischen Konsistenz für unendlich viele Daten ergänzt.

Datenbasierte Regelung nichtlinearer Systeme durch polynomielle Approximation

Basierend auf dem vorherigen Kapitel und polynomiellen Approximationsmethoden, führt Kapitel 4 eine neue datenbasierte Systembeschreibung für nichtlineare Systeme ein. Diese ermöglicht eine Systemanalyse und einen Entwurf von Zustandsrückführungen mittels SDPs trotz nichtlinearer Dynamik. Hierzu werden bekannte Schranken für den Approximationsfehler polynomieller Interpolationen mit einer datenbasierten Menge von Polynomen kombiniert, die das unbekannte Interpolationspolynom enthält. Speziell betrachten wir TPs und HPs, wobei letztere zusätzliche Freiheitsgrade besitzen, um die Polynomapproximation zu verbessern. Um den Konservatismus der Approximation weiter zu reduzieren, können mehrere Polynomapproximationen aus den Daten bestimmt werden, wobei die einzelnen Polynome jeweils die Dynamik nur für eine Untermenge des Arbeitsbereiches approximieren müssen. Außerdem ermöglicht die polynomielle Systembeschreibung die Bestimmung von Eingangs-/Ausgangseigenschaften durch SDPs. Zusätzlich zu Kapitel 3 betrachten wir hier den Entwurf von Zustandsrückführungen mit Performancekriterien und den Fall, dass das Rauschen Gaußverteilt ist. Die erarbeiteten Methoden werden in numerischen Beispielen, aber auch experimentell für ein Zweitank getestet.

Zusammenfassend etabliert diese Doktorarbeit eine Methode zur Verifikation von Systemeigenschaften und zum Entwurf von Zustandsreglern für nichtlineare Systeme aus verrauschten Eingangs-/Zustandsdaten. Obwohl die zugrundeliegende Dynamik nichtlinear und die Daten verrauscht sind, ermöglicht diese Methode garantierte Rückschlüsse auf die zugrundeliegende Dynamik und die Anwendung von SDPs, um ein weites Spektrum von regelungstechnischen Problemstellungen effektiv zu lösen.

Chapter 1

Introduction

1.1 Motivation

The application of classical control methods relies on a mathematical model of the underlying dynamics. To this end, the system behavior over time is characterized by difference or differential equations of the internal states, control inputs, external disturbances, and measured outputs. A common avenue to determine these models are first principles, e.g., Newton's laws of motion for mechanical system or Kirchhoff's circuit laws for electrical systems. However, obtaining a sufficiently precise model from first principles can be cumbersome due to the increasing complexity of systems in engineering, for instance, soft robots [33], autonomous cars [78], and biological systems [173]. Therefore, their successful application is jeopardized due to the requirement of expert knowledge, a priori model simplifications, as well as in many cases being more time-consuming than the controller design itself. At the same time, measured input-output trajectories of the system are often available in storage or can easily be gathered by exciting the system in experiments. For these reasons, alternative methods for deriving controllers from input-output measurements have been studied in the context of data-driven control [74]. On the one hand, indirect data-driven control methods require the two-step procedure of first deriving a data-based system representation and subsequently applying model-based control techniques. On the other hand, direct data-driven control refers to directly deducing controllers from data.

System identification techniques together with a model-based controller design constitute a well-established indirect data-driven control method, where a model can be obtained from data by a large bundle of different approaches [14, 92, 116]. To ensure guarantees for the resulting closed loop, a model together with a bound for the estimation mismatch need to be deduced. However, determining the mismatch in the scenario of finite noisy data is challenging even for LTI systems [161]. For stochastic noise, promising directions include the works [102, 122, 147], where the study of non-asymptotic guarantees still requires strong assumptions on the data and noise. For instance, [122] asks for Gaussian iid noise and input as well as a zero initial condition. In case of bounded deterministic noise, set-membership identification [108] strives to identify a model and to estimate an error bound from the set of all LTI systems which could generate the measured trajectories. For identifying nonlinear systems, the set-membership methodology is also applicable by means of a Lipschitz approximation [107], but may lead to complex models. Furthermore, the literature covers approaches tailored for special classes of nonlinear systems [184], requiring an appropriate function basis [148], or using deep neural networks [93]. While these approaches can perform well in practice, they rarely come with guarantees for their approximation error.

Avoiding the intermediate step of identifying a model and bounding its mismatch, direct data-driven control techniques include PID control [191], adaptive control [15], iterative feedback tuning [73], virtual reference feedback tuning [37], reinforcement learning [32], unfalsified control [135], and subspace-based LQG-control [56]. We also refer to the survey [74] and the references therein. Despite their empirical success, most of these approaches do not provide theoretical guarantees, and thus may result in unpredictable closed-loop behavior [130]. However, the rising interest in reliable data-driven control approaches since the publication of the survey [74] have led to new methods establishing a comprising framework for data-driven control of LTI systems. More specifically, the behavioral viewpoint led to the fundamental lemma [183], which achieves a simple algebraic condition with a Hankel matrix of a measured trajectory to parametrize all possible trajectories of the underlying system. Therefore, this non-parametric system representation has inspired direct data-driven state feedback control [52], predictive control [18, 47], and more [101]. Besides Willems' fundamental lemma, data informativity [177, 178] specifies when noisy data enable to draw conclusions concerning dissipativity, stabilizability, etc., which requires in general less data than system identification. Equivalently to the set-membership methodology [60], the data informativity framework is also based on the set of all systems explaining the noisy data. The combination of set membership and modern robust control techniques is exploited in [41] to render an unknown LTI system (super-)stable using noisy input-output data by linear programming.

An intermediate approach between system identification and a controller synthesis directly from data is the verification of control-theoretic input-output properties, for instance, dissipativity [182] or IQCs [105]. On the one hand, the system behavior is expressed by the verified properties, which enables a controller synthesis by well-studied feedback laws [53, 80, 137, 164, 187], e.g., the small-gain theorem or the interconnection of passive systems. The obtained controllers attain closed-loop guarantees for stability and robustness. Thus, determining system properties can be beneficial compared to identifying a complex model not suitable for a controller synthesis. On the other hand, these properties offer valuable insights into the system. For instance, NLMs according to [3, 4, 54, 65, 118, 145, 149] quantify the strength of the nonlinearity of a dynamical system, and thus give an intuition whether a linear controller design can be successful. Simultaneously, determining these NLMs includes the calculation of the 'best' linear approximation of the nonlinear input-output behavior, which might be preferable as a linear surrogate model over the Jacobi-linearization. Similar to direct data-driven control, most works on data-driven system analysis with theoretic guarantees are restricted to LTI systems including iterative sampling schemes [83, 179], the behavioral viewpoint [103, 131], and the data informativity framework [81, 177].

While Willems' fundamental lemma and the data-informativity framework build a fruitful basis for developing data-driven control methods with rigorous guarantees for LTI systems, data-driven control of nonlinear systems comes with additional challenges. First, the nonlinearity of the unknown dynamics precludes guarantees for the identification with deterministic error bounds from finitely many samples. Therefore, additional assumptions are required, e.g., a function basis containing the dynamics recovered from first principles. Consequently, the data-driven inference relies not only on the data's informativity but also on the accuracy of the prior knowledge. This corresponds to a second challenge. Third, even if a suitable function basis is available analyses such as the estimation of the region of attraction or a controller design with performance guarantees involve nonconvex optimization. This usually prevents an efficient numerical solution. Summarizing, due to the mentioned challenges, many data-driven control approaches of nonlinear systems lack rigorous guarantees for finite noisy data, call for nonconvex optimization, or require an appropriate function basis containing the underlying dynamics.

These drawbacks motivate the data-driven control framework for nonlinear systems presented in this thesis. Within this framework, we can solve control-related problems that are demanding even under precise knowledge of the nonlinear dynamics. To this end, we introduce a novel system representation consisting of a set membership for polynomial approximation, which can be obtained from noisy data. Thereby, we can determine system-theoretic properties, e.g., dissipativity and NLMs, and design state-feedback controllers without identifying a model and estimating its error. Due to the polynomial description of the system representation, the verification of system properties and the design of controllers satisfying performance criteria boil down to effectively solvable SDPs.

1.2 Related work

Within the challenging research field of data-driven control for nonlinear systems, many research efforts have been made and a variety of approaches has been developed. For the sake of presentation, we concentrate on data-based methods providing guarantees. Note that this section is taken in parts literally from [TM10][1].

Data-driven control techniques with guarantees include nonlinear adaptive control [13]. However, the successful application of adaptive control calls for persistently exciting closed-loop trajectories, which can be challenging in some applications. The field of learning-based model predictive control [71] provides a flexible framework with rigorous guarantees, but requires nonconvex optimization during runtime. Probabilistic guarantees are ensured in reinforcement learning with safety guarantees [23] and stability verification [89] using scenario optimization [36]. Both ask for nonconvex optimization for general nonlinear systems and iid sampled data. Furthermore, neural networks are employed in [112] for stability guarantees by learning the dynamics, the controller, and a Lyapunov function at the same time. However, an obscure approximation error of the neural networks are required together with a large amount of samples. Lipschitz approximation of nonlinear systems has been leveraged for a wide range of data-driven system analysis and control problems. To resolve the requirement of nonconvex optimization, a stream of research has investigated data-based system representations for nonlinear systems suitable for a controller design by convex optimization formulated as SDPs. Due to their connection to our framework, data-driven control design by Lipschitz approximation and SDPs are discussed in more detail in the following.

[1] T. Martin, T. B. Schön, and F. Allgöwer. “Guarantees for data-driven control of nonlinear systems using semidefinite programming: A survey.” In: *Annual Reviews in Control* 56 (2023), p. 100911.

Lipschitz approximation

The key idea is that, under a known Lipschitz constant of an unknown function, we can construct a cone for each sample point containing the graph of this function. Thus, the intersection of cones from multiple samples constitutes a non-parametric envelope around the function. For that reason, this representation circumvents the need to first choose a parametrized model and can be utilized for various control-related problems as summarized in the sequel.

In nonlinear set-membership identification [107, 114], an optimal envelope containing the to-be-estimated nonlinear dynamics is obtained from the non-parametric representation. Thus, a guaranteed robust prediction of the system behavior is possible. This can be leveraged, for instance, in learning-based model predictive control [71]. In addition, the set of all feasible functions enables to derive an estimation of the unknown function together with its worst-case identification error. In machine learning, equivalent results have been presented as Kinky inference for learning and predicting unknown nonlinear functions [34] (Chapter 4). Therein, also extensions for employing local data [29] and Hölder continuous functions are suggested. Due to their relevance, a manifold of validation procedures has been proposed for estimating Lipschitz or Hölder constants of unknown nonlinear functions from data [35, 107].

In the context of data-driven system analysis, the Lipschitz constant together with input-output tuples of the input-output mapping of an unknown nonlinear system are employed in [113] to infer the $\mathcal{L}_2$-gain, the shortage of passivity, and a cone. This idea is further analyzed regarding the sample complexity [146] and more general dissipation inequalities [TM6, 132]. All these approaches require to estimate the covering radius for the input space, which measures how densely the input space is sampled. Alternatively, [TM5] directly exploits local Kinky inferences of the input-output mapping to determine its gain. There, not only data from storage are employed but also an iterative sampling scheme is proposed to reduce the sample complexity.

Besides learning-based model predictive control, the set membership from Lipschitz approximation has also been applied for online control approaches [157], model reference adaptive control [34] (Section 4.4), and offline direct controller design [119]. In contrast to offline approaches, an online approach has the advantage that also currently measured data can be incorporated to refine the control performance but requires more computation during runtime.

The downside of Lipschitz approximations usually is the excessive demand of samples. For instance, the inference of system properties requires thousands of experiments [113]. This makes the approach rather impractical. Moreover, the obtained insights on the system properties only hold for a data-dependent finite horizon, whereas the feedback laws [53, 80, 137, 164, 187] call for an arbitrarily long horizon. The data-based system representation proposed in Chapter 4 generalizes Lipschitz approximation towards data-driven polynomial approximation, and thereby resolves these drawbacks.

State-feedback design based on SDPs for nonlinear systems

Gaussian processes and kernel ridge regression constitute a flexible framework to approximate nonlinear functions in machine learning and nonlinear dynamics in system identification. Both regression methods include the possibility for incorporating prior knowledge and inherent uncertainty measures to derive guarantees for data-driven control. However, the obtained system representation is often strongly nonlinear due to nonlinear kernel functions. To deal with this nonlinearity, [163] presents a controller design by feedback linearization and [38] by backstepping. However, the former requires that the input enters the dynamics via an invertible, known, and state-depended matrix. The latter calls for a structure of the dynamics typical for backstepping. Based on [59], the same authors of [58] obtain a linear sector for the nonlinear parts of the dynamics from a Gaussian process to apply linear robust control afterwards. Instead of bounding the mismatch of the kernel regression by a sector, [22] directly computes the Jacobian linearization of the nonlinear Gaussian-process-model around an equilibrium point for a linear robust controller design. Alternatively, [77] suggests to stabilize the linear part of a kernel regression, while approximately cancelling its nonlinearity by an SDP as in [51]. The work [55] proposes to use polynomial kernels yielding a polynomial regression model and a polynomial sector for the approximation error. Thus, a system analysis and a controller design by SOS techniques are possible. Related to these results, [133] determines passivity properties via Gaussian process optimization, which then facilitates an active sampling scheme to improve the data-driven inference of the system property. In summary, these works require a certain structure of the dynamics or conclude on a polynomial approximation of the dynamics using Gaussian processes or kernel ridge regression. In contrast, we present in Chapter 4 a direct manner to infer polynomial approximations from data, which thus circumvents the regression error.