Eigendamage: An Eigendeformation Model for the Variational Approximation of Cohesive Fracture

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Chapter 1

Introduction

The work presented in this thesis is devoted to an approximation scheme for the variational theory of fracture in ductile materials. In contrast to brittle materials, which respond to very small displacements by elastic deformations and subsequently develop cracks, in ductile materials a plastic deformation takes place before rupture. Since plasticity plays a significant role in the fracture of ductile materials, the occurrence of plastic deformations has to be taken into account in an accurate description of the process of ductile crack growth. Models which are applied in the theory of brittle fracture are therefore not suitable to predict the crack propagation of ductile materials. Models which can deal with the nonlinear zone near the crack tip due to plasticity, are the cohesive (zone) models. They are therefore widely used in fracture mechanics to describe ductile fracture processes.

In this thesis, we propose and investigate a variational approximation scheme for cohesive models in the case of antiplane shear. The variational model studied in this work is described by functionals depending on a small parameter and on two fields: the displacement field and an eigendeformation field, where the latter one describes the permanent deformations that may occur in the body. The focus lies on the asymptotic behavior of this model in the static case. The rigorous mathematical investigation of the proposed model relies on the notation of Γ-convergence. From a mechanical point of view this approach is based on the notion of eigendeformation.

Before presenting the model that we consider, and in order to explain our main results in detail, we give an introductory overview to the variational modeling and variational approximation of fracture in fracture mechanics. We thereby highlight the main mathematical differences between energies used in brittle fracture and in cohesive fracture. In addition, we briefly discuss possible applications of these energies in the field of image processing.

Variational modeling of fracture and other problems related to image processing

The first formulation of (brittle) fracture problems in a variational setting was proposed by Francfort and Marigo in [43]. Based on the renowned work of Griffith [49] which shows that the propagation of a fracture depends on the balance between the elastic energy released when the crack grows and the energy dissipated to enlarge the crack or to produce a new one, Francfort and Marigo introduced an energy functional (of Griffith type) comprising of a bulk term corresponding to the stored elastic energy and a surface term modelling the fracture energy. For further references in particular of subsequent works on this topic see [18].

From a mathematical point of view, determining energy minimizers of such functionals leads to a big class of variational problems, the so-called free discontinuity problems. This term was first introduced by De Giorgi and Ambrosio in [36] to denote problems involving competitive volume energies, concentrated on d-dimensional sets, and surface energies, depending on $(d-1)$-dimensional sets.

There are many mathematical problems arising not only from fracture mechanics but also from image processing, computer vision theory, liquid theory and phase transition, which are characterized by a competition of bulk and surface energies and can be therefore formulated as free discontinuity problems (cf., e.g., [8, 20, 33]).

The first problem that was, however, studied within this framework was the Mumford-Shah model which is applied in image processing to detect the contours of an object in a black-and-white image. By means of this classical example we briefly discuss the existence theory of minimizers for free discontinuity problems in what follows. In the course of this, we also give a short description of the model in the context of image segmentation. For more details about the Mumford-Shah model and image segmentation see, e.g., [44] and [11, 58, 67].

Image segmentation in principle deals with the problem of dividing an image into multiple parts in order to identify objects or other relevant information of this image. In the Mumford-Shah model that was proposed by Mumford and Shah in [59], there are an open set Ω in the plane and a grey-scale function $g : \Omega \to [0,1]$. The aim is to determine a pair (u, K), where $K \subset \Omega$ is a compact set representing the set of contours of the object in the image and $u \in C^1(\Omega \setminus K)$ is a smooth approximation of g outside of K. This pair is determined by minimizing the functional

$$\mathcal{MS}(u,K) = \alpha \int_{\Omega\setminus K} (u-g)^2 \,\mathrm{d}x + \int_{\Omega\setminus K} |\nabla u|^2 \,\mathrm{d}x + \beta \mathcal{H}^1(K). \tag{1.1}$$

The first term of (1.1) forces u to be close in an integral sense to the original input image g, while the second term penalizes strong variation of u. The one-dimensional term $\mathcal{H}^1(K)$ measures the length of the contours and it is used to prevent oversegmentation. In addition, α and β are scaling and contrast parameters. Note that the set K is not assigned a priori and it is in general not a boundary. Any minimizer (u, K) of $\mathcal{MS}$ is called an optimal pair. In order to show the existence of an optimal pair, one usually applies the direct method of the calculus of variation, which in particular requires the lower semicontinuity of $\mathcal{MS}$. The main problem (with the lower semicontinuity of $\mathcal{MS}$) comes along with the term $\mathcal{H}^1(K)$ which is in general not lower semicontinuous with respect to the Hausdorff convergence unless some additional assumptions on the approximating sequences are made. Therefore, weak formulations of the problem in a suitable functional setting are needed. When following the idea of De Giorgi to interpret the set K as the closure of the set of jump points J_u of the function u and taking into account the structure of the energy functional, it is possible to reformulate the Mumford-Shah problem in the weak functional framework of the space SBV of special functions with bounded variation introduced by De Giorgi and Ambrosio [36].

Before going into details, we recall some facts on functions of bounded variation and on special functions of bounded variation. Hereafter, let Ω be an open subset of $\mathbb{R}^d$ with Lipschitz boundary. A function u lies in $BV(\Omega)$ if $u \in L^1(\Omega)$ and its distributional derivative Du is a finite Radon measure. Most important in the theory of BV-functions is the decomposition of the measure Du in

$$Du = \nabla u \mathcal{L}^d + D^s u,$$

where ∇u denotes the density of Du with respect to the d-dimensional Lebesgue measure and $D^s u$ is the singular part. Moreover, the measure $D^s u$ can be split as

$$D^s u = D^j u + D^c u,$$

where $D^j u$ is the jump part and $D^c u$ is the Cantor part. Details about BV-functions can be found in [8, 41]. We only want to mention that the measure $D^j u$ is of the form $[u]\,\nu_u\,\mathcal{H}^{d-1} \llcorner J_u$, where J_u is the $\mathcal{H}^{d-1}$-rectifiable jump set, $\nu_u : J_u \to \mathbb{S}^{d-1}$ is its normal and $[u]$ is the jump function, which is the difference between the traces of u on the two sides of a jump. A function u belongs to $SBV(\Omega)$ if $u \in BV(\Omega)$ and $D^s u = D^j u$.

The weak formulation of the Mumford-Shah functional is given by

$$\mathcal{MS}_w(u) := \alpha \int_\Omega (u-g)^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \beta \mathcal{H}^1(J_u) \quad \text{for } u \in SBV(\Omega). \tag{1.2}$$

The existence of a minimizer $u \in SBV(\Omega)$ of the weak formulation in (1.2) can be shown by the compactness and lower semicontinuity of the space $SBV(\Omega)$ with respect to weak*-convergence (first proven by Ambrosio in [6]). Finally, one can indeed show that any minimizer of $\mathcal{MS}_w$ provides an optimal pair $(u, \bar{J}_u)$ for $\mathcal{MS}$.

Inspired by this example, weak formulations of functionals of Griffith type were proposed (cf. [43] and [18]). In the static case one obtains an energy functional defined by

$$\mathcal{G}_w(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \kappa \mathcal{H}^1(J_u) \quad \text{for } u \in SBV(\Omega). \tag{1.3}$$

that the variational formulation for Griffith theory of brittle fracture (1.3) is equal, up to constants, to the weak formulation of the Mumford-Shah problem in (1.7).

A first approximation by Γ-convergence of (1.7) was given by Ambrosio and Tortorelli in [9] by following an earlier idea developed by Modica and Mortola [57], who approximated the perimeter functional by elliptic functionals. In the approach of Ambrosio and Tortorelli, the set J_u is replaced by an auxiliary variable v that approximates the function $1 - \chi_{J_u}$. In this approximation, a family of elliptic functionals of the following form is considered

$$AT_\varepsilon(u, v) = \int_\Omega v^2 |\nabla u|^2 \, \mathrm{d}x + \frac{\beta}{2} \int_\Omega \left(\varepsilon |\nabla v|^2 + \frac{1}{\varepsilon}(1 - v)^2 \right) \mathrm{d}x, \tag{1.8}$$

defined for $u, v \in W^{1,2}(\Omega)$ and $0 \leq v \leq 1$. In [9], it was proved that the family $(AT_\varepsilon)_\varepsilon$ Γ-converges as $\varepsilon \to 0^+$ with respect to the $L^1(\Omega) \times L^1(\Omega)$-topology to the functional F defined by

$$F(u, v) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \beta \mathcal{H}^1(J_u) \tag{1.9}$$

if $u \in SBV(\Omega)$ and $v = 1$ a.e. on Ω, and $F(u, v) = +\infty$ otherwise. Note that minimizing (1.9) is obviously equivalent to minimize (1.7). See [42] for the generalization of this result to the case of vector valued functions.

Although the functional AT_ε was originally introduced for a variational approach to image segmentation, it is nowadays also used in fracture mechanics to describe a damage model (cf. [62, 63]). In the latter application the auxiliary variable v in (1.8) is a measure of the damage state of the corresponding material points and it is therefore called damage variable. In other words, the material shows an elastic response in regions where $v = 1$. As $v \to 0$ the process of degradation of the material increases up to the maximum possible damage, which is reached at $v = 0$. In the context of damage models, the result of Ambrosio and Tortorelli [9] was extended in various directions in the setting of brittle fracture [34, 50], for instance, to the setting of linearized elasticity [27, 28, 51]. In addition, damage models have also been studied to approximate cohesive energies like Barenblatt's energy and Dugdale's energy (cf. [3, 29, 35]).

A completely different approach is given by non-local approximations. In [23], Braides and Dal Maso proved the first result in this direction. They considered a family of non-local functionals of the form

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon} \int_\Omega f \left(\varepsilon \fint_{B_\varepsilon(x) \cap \Omega} |\nabla u(t)|^2 \, \mathrm{d}t \right) \mathrm{d}x \tag{1.10}$$

defined for $u \in W^{1,2}(\Omega)$. Here, $B_\varepsilon(x)$ is the open ball with center x and radius ε and $\fint_B$ indicates the average on the set B. The function $f : [0, +\infty) \to [0, +\infty)$ is non-decreasing, continuous and such that

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 1 \quad \text{and} \quad \lim_{t \to +\infty} f(t) = \frac{\beta}{2}.$$

The functionals $\mathcal{F}_\varepsilon$ are non-local in the sense that their energy density at a point $x \in \Omega$ depends on the behaviour of u on the whole set $B_\varepsilon(x) \cap \Omega$. In [23], it was shown that the family $(\mathcal{F}_\varepsilon)_\varepsilon$ Γ-converges as $\varepsilon \to 0^+$ with respect to $L^2(\Omega)$-topology to the Mumford-Shah functional (1.7).

By considering non-local functionals where f is replaced by a suitable function f_ε in (1.10), functionals with more general surface energies can be approximated in the sense of Γ-convergence (see [25]). Later this result was generalized by Cortesani and Toader [31] in the setting of functions of bounded variation and later on by Negri [61] in the space $SBD(\Omega)$ of special functions with bounded deformation (and with more general hypotheses on the functions f_ε) for non-local operators depending on more general convolution kernels. For completeness we also recall that another approach based on a different type of non-locality and finite differences was used by Gobbino to approximate the Mumford-Shah functional in [47]. See also [48] for the approximations of functionals with more general surface energies. The main difference of this approximation is that the convolution term is outside of the integrand f and that ∇u is replaced by a finite difference.

To conclude the discussion about non-local approximations, let us stress that Lussardi and Vitali [55] (see also [54] for the corresponding result in one dimension) studied the asymptotic behavior of functionals $\mathcal{F}_\varepsilon$ defined on $L^1(\Omega)$ by

$$\mathcal{F}_\varepsilon(u) := \frac{1}{\varepsilon} \int_\Omega f\left(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(t)|\,\mathrm{d}t\right) \mathrm{d}x$$

if $u \in W^{1,1}(\Omega)$, and $\mathcal{F}_\varepsilon(u) = +\infty$ otherwise. Here, $f : [0,+\infty) \to [0,+\infty)$ is non-decreasing, strictly concave and twice continuously differentiable and satisfies

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 1.$$

They proved that $(\mathcal{F}_\varepsilon)_\varepsilon$ Γ-converges as $\varepsilon \to 0^+$ with respect to the $L^1(\Omega)$-topology to the functional $\mathcal{F}$ which is defined on $L^1(\Omega)$ by

$$\mathcal{F}(u) := \int_\Omega |\nabla u|\,\mathrm{d}x + \int_{J_u} \theta(|[u]|)\,\mathrm{d}\mathcal{H}^{d-1} + |D^c u|(\Omega)$$

if $u \in L^1(\Omega)$ and its truncations are in $BV(\Omega)$, and $\mathcal{F}(u) = +\infty$ otherwise. The surface density $\theta : [0,+\infty) \to [0,+\infty)$ is given by

$$\theta(s) := 2\int_0^1 f\left(\frac{\omega_{d-1}}{\omega_d} s \left(\sqrt{1-t^2}\right)^{d-1}\right) \mathrm{d}t \quad \text{for all } s \in [0,+\infty), \tag{1.11}$$

where ω_d and ω_{d-1} denote the volume of the d-dimensional ball in $\mathbb{R}^d$ and the volume of the $(d-1)$-dimensional ball in $\mathbb{R}^{d-1}$(with $\omega_0 = 1$), respectively. This result was generalized by Lussardi and Magni [53] for more general non-local operators by considering convolutions between the gradient and not radially symmetrical kernels.